Table 1 Tapered beams: frequency amplitude relationships

Beam	Assumed displacement distribution for w	ω_{NL}^2/ω_L^2	Linear frequency parameter $\rho A_c \omega_L^2 l^4 / E I_c$	Breadth taper λ	Breadth taper μ	Depth taper λ	Depth taper μ
Simply supported	$a\cos\frac{\pi\xi}{2}$	$1-\frac{3}{16}\beta^2\frac{\alpha}{Ln(1-\alpha)}\frac{1}{1+\lambda}$	$\pi^4\frac{1+\lambda}{1+\mu}$	$(\frac{2}{\pi^2}-\frac{1}{2})\alpha$	$(\frac{2}{\pi^2}-\frac{1}{2})\alpha$	$\alpha(\frac{6}{\pi^2}-\frac{3}{2})+\alpha^2(1-\frac{6}{\pi^2})+\alpha^3(-\frac{12}{\pi^4}-\frac{1}{4}+\frac{3}{\pi^2})$	$(\frac{2}{\pi^2}-\frac{1}{2})\alpha$
	$a(1-\frac{6}{5}\xi^2+\frac{1}{5}\xi^4)$	$1-\frac{3}{16}\beta^2\frac{\alpha}{Ln(1-\alpha)}\cdot\frac{1}{1+\lambda}\frac{1088\times 17}{875\times 21}$	$\frac{48\times 63}{31}\frac{1+\lambda}{1+\mu}$	$-\frac{5}{16}\alpha$	$-\frac{113\times 21}{7936}\alpha$	$-\frac{15}{6}\alpha+\frac{3}{7}\alpha^2-\frac{5}{64}\alpha^3$	$-\frac{113\times 21}{7936}\alpha$
Clamped	$\frac{a}{2}(1+\cos\pi\xi)$	$1-\frac{3}{16}\beta^2\frac{\alpha}{Ln(1-\alpha)}\cdot\frac{1}{4(1+\lambda)}$	$\frac{16\pi^4}{3}\frac{1+\lambda}{1+\mu}$	$-\frac{\alpha}{2}$	$(\frac{8}{3\pi^2}-\frac{1}{2})\alpha$	$-\frac{3}{2}\alpha+(1+\frac{3}{2\pi^2})\alpha^2-\alpha^3(\frac{1}{4}+\frac{3}{4\pi^2})$	$(\frac{8}{3\pi^2}-\frac{1}{2})\alpha$
	$a(\xi^2-1)^2$	$1-\frac{3}{16}\beta^2\frac{\alpha}{Ln(1-\alpha)}\cdot\frac{1}{(1+\lambda)}\cdot\frac{512}{105\times 21}$	$504\frac{1+\lambda}{1+\mu}$	$-\frac{5}{8}\alpha$	$-\frac{63}{256}\alpha$	$-\frac{15}{8}\alpha+\frac{11}{7}\alpha^2-\frac{15}{32}\alpha^3$	$-\frac{63}{256}\alpha$

Table 2 ω_{NL}/ω_L of a tapered beam for $\alpha=0.4$

Amplitude ratio $\beta=a/r_c$	Simply supported beam				Clamped beam			
	Breadth taper		Breadth taper		Breadth taper		Depth	
	trigonometric[a]	polynomial[b]	trigonometric[a]	polynomial[b]	trigonometric[c]	polynomial[d]	trigonometric[c]	polynomial[d]
0.1	1.0008	1.0008	1.0010	1.0010	1.0002	1.0002	1.0004	1.0005
0.2	1.0033	1.0034	1.0042	1.0042	1.0009	1.0009	1.0017	1.0018
0.4	1.0132	1.0134	1.0166	1.0169	1.0037	1.0036	1.0066	1.0074
0.6	1.0296	1.0300	1.0370	1.0375	1.0082	1.0081	1.0149	1.0165
0.8	1.0520	1.0527	1.0649	1.0658	1.0146	1.0144	1.0263	1.0291
1.0	1.0801	1.0812	1.0997	1.1011	1.0227	1.0225	1.0408	1.0451
2.0	1.2910	1.2944	1.3354	1.3601	1.0879	1.0871	1.1545	1.1702
3.0	1.5811	1.5875	1.6981	1.7065	1.1887	1.1870	1.3224	1.3532

[a] $w=a\cos\pi\xi/2$.
[b] $w=a(1-6/5\,\xi^2+1/5\,\xi^4)$.
[c] $w=a/2\,(1+\cos\pi\xi)$.
[d] $w=a(\xi^2-1)^2$.

The previous analysis is repeated for a simply supported beam for a polynomial distribution and a clamped beam with trigonometric and polynomial distributions for breadth and depth tapers. The results are summarized in Table 1.

Numerical Results

Table 2 presents ω_{NL}/ω_L for simply supported and clamped beams with breadth and depth tapers for a typical taper parameter $\alpha=0.4$. These results are obtained with the solutions presented in Table 1. Table 2 demonstrates good agreement between trigonometric and polynomial solutions for all cases. Further, it is noted that the nonlinearity is always of hardening type and, as expected, the nonlinearity is severe for beams with depth taper compared to beams with breadth taper.

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Anisotropy and Shear Deformation in Laminated Composite Plates

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Introduction

It has long been recognized that transverse shear flexibility and material anisotropy (nonorthotropy) can play an important role in reducing the effective stiffness of laminated composite plates. However, most of the published results for statically loaded plates are for simply supported edges and are limited to comparisons of the maximum deflections and stresses obtained by refined theories (which include transverse shear flexibility) with those obtained by classical laminated plate theory.[1-3] To the authors' knowledge, no quantitative measures have been used which account for the effects of the different lamination and geometric parameters on the transverse shear flexibility and material anisotropy of composite plates. The present Note summarizes some of the results of a

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Table 1 Values of the composite shear correction factors for graphite-epoxy plates with $E_L/E_T=40$, $G_{LT}/E_T=0.5$, $G_{TT}/E_T=0.4$, $\nu_{LT}=0.25$

NL	$k^2_{(1)},k^2_{(2)}$	NL	$k^2_{(1)}$	$k^2_{(2)}$
1	0.8333	3	0.8173	0.5312
2	0.6156	5	0.8658	0.5838
4	0.6422	7	0.8709	0.6697
10	0.7902	9	0.8684	0.7127

recent study which focuses on this problem. It aims at investigating the effect of variations in the geometry, lamination parameters, and boundary conditions on the significance of shear deformation, and degree of anisotropy (nonorthotropy) of composite plates.

The analytical formulation is based on a two-dimensional shear deformation plate theory with the effects of anisotropic material behavior and bending-extensional coupling included.[3] For multilayered plates the composite shear correction factors introduced in Refs. 4 and 5 are included. The use of these composite correction factors was found to result in close agreement with three-dimensional elasticity solutions.[6] Numerical solutions of the shear deformation theory were obtained using a higher-order shear flexible quadrilateral (rectangular) finite element model. The element had a total of 80 degrees of freedom and the shape functions used in approximating each of the displacement and rotation components consisted of products of first-order Hermitian polynomials.[7] Such a finite element model was shown to give highly accurate results for the response characteristics of the plate.

Quantitative Measures for Shear Deformation and Degree of Anisotropy (Nonorthotropy)

The significance of shear deformation and degree of anisotropy in composite plates depends, basically, on 2 groups of parameters, namely: 1) Intrinsic parameters which include lamination and geometric parameters (e.g., fiber orientation, number and stacking of the layers, degree of orthotropy of individual layers, thickness, and aspect ratio of the plate). 2) Extrinsic parameters which include loading and boundary conditions.

A quantitative measure for the transverse shear deformation, which accounts for the aforementioned effects, is provided by the ratio of the transverse shear strain energy to the total strain energy of the plate, U_{sh}/U. As a measure for the degree of anisotropy (nonorthotropy) of the plate, the ratio of the contribution of the anisotropic terms to the total strain energy of the plate is used, U_a/U. Herein, the anisotropic terms refer to the elastic coefficients which are characteristic of general anisotropic (nonorthotropic) plates, namely A_{16}, A_{26}, B_{16}, B_{26}, D_{16}, D_{26}, and C_{45}.[8] Since all of these coefficients vanish for orthotropic and isotropic plates and are nonzero only for anisotropic plates, it seems reasonable to take their contribution to the strain energy of the plate U_a/U as a measure of its degree of anisotropy (nonorthotropy). Other measures have been suggested and used in the literature, such as fiber orientation angle and the ratio of the bending stiffnesses D_{16}/D_{11} and D_{26}/D_{11}.[1,8] However, these measures do not account for the aforementioned intrinsic and extrinsic plate parameters.

Since anisotropy is a directional property, the quantitative measure U_a/U depends on the particular choice of the coordinate system. On the other hand, the transverse shear deformation measure, U_{sh}/U, is invariant with respect to coordinate transformations.

Numerical Results

Since closed form solutions cannot be obtained, in general, for anisotropic plates, the significance of shear deformation

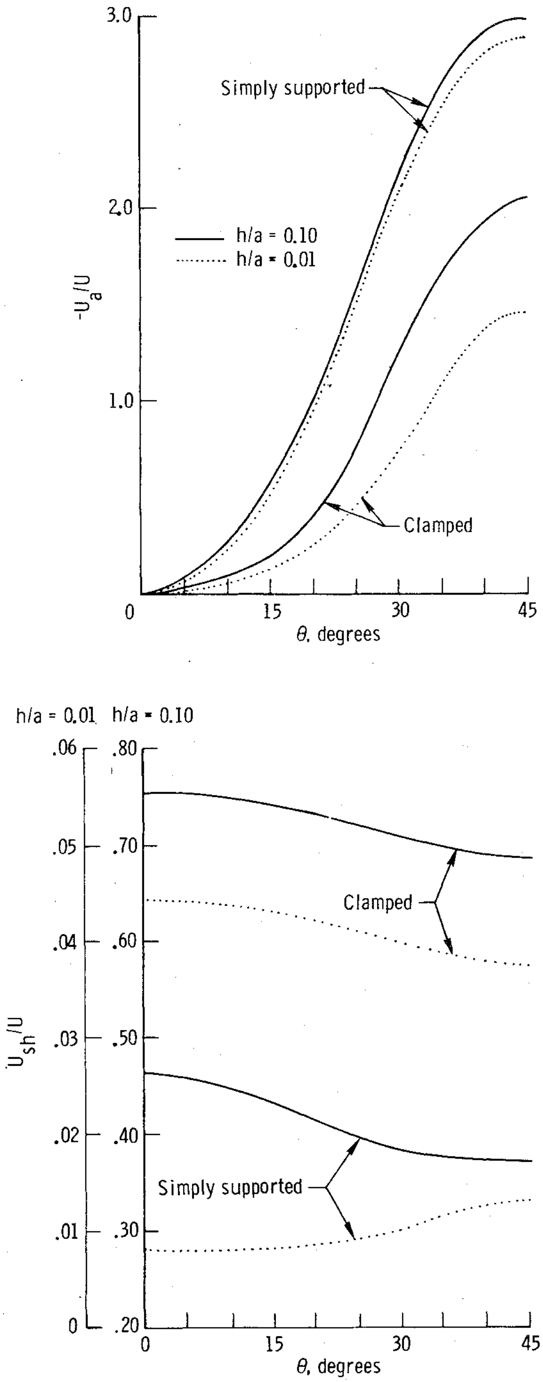


Fig. 1 Effect of fiber orientation angle θ on significance of shear deformation and degree of anisotropy. Single-layer graphite-epoxy square plates subjected to uniform loading.

and degree of anisotropy was studied by solving numerically a large number of problems covering a wide range of lamination and geometric parameters of the plate. Single layer square anisotropic plates as well as laminates having both symmetric and skew-symmetric orientation with respect to the middle plane were considered. The fiber orientation of the different laminas alternate between $+\theta$ and $-\theta$ with respect to the x_1-axis ($0<\theta\leq 45°$). In the symmetrical laminates the $+\theta$ layers were at the outer surfaces of the laminate. The total thickness of the $+\theta$ and $-\theta$ layers in each laminate were the same.

The material characteristics of the individual layers were taken to be those typical of high-modulus graphite-epoxy composites, namely $E_L/E_T=40$, $G_{LT}/E_T=0.5$, $G_{TT}/E_T=0.4$, $\nu_{LT}=\nu_{TT}=0.25$ where subscript L refers to the direction of fibers and subscript T refers to the transverse direc-

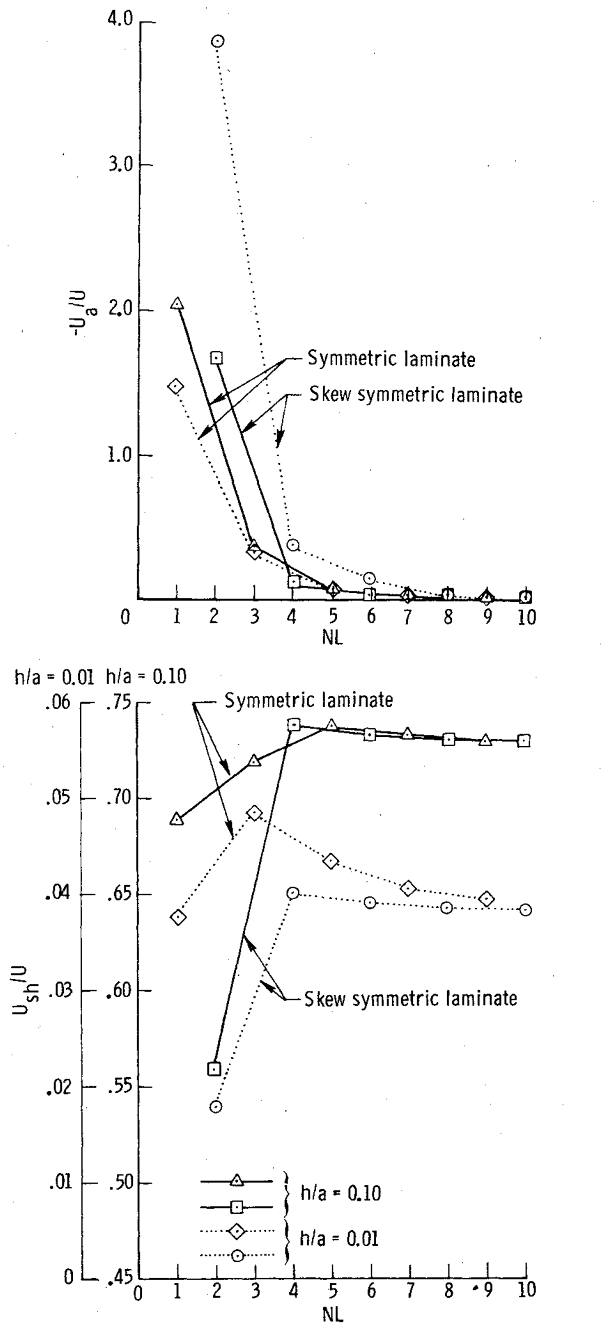


Fig. 2 Effect of the number of layers *NL* on significance of shear deformation and degree of anisotropy. Clamped graphite-epoxy square plate subjected to uniform loading, $\theta = 45°$.

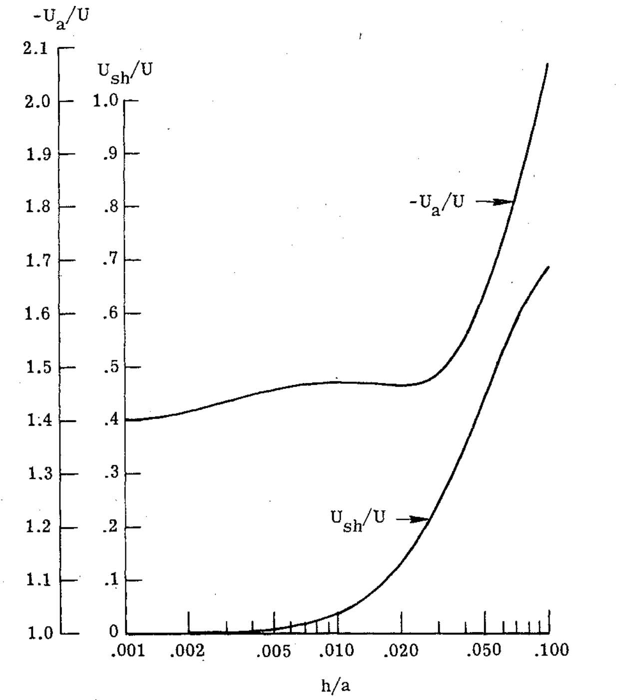


Fig. 3 Effect of thickness ratio h/a on significance of shear deformation and degree of anisotropy. Single-layer clamped graphite-epoxy square plate subjected to uniform loading, $\theta = 45°$.

tion, ν_{LT} is the major Poisson's ratio. Both simply supported and clamped plates are considered. The composite shear correction factors for the multilayered plates considered in the present study are given in Table 1.

Three parameters were varied, namely the fiber orientation angle θ, the number of layers *NL* and the thickness ratio of the plate h/a, where h and a are the thickness and side length of the plate. The angle θ was varied between 0 and 45°. *NL* was varied between 1 and 10 and h/a was varied between 0.001 and 0.10. Typical results showing the effect of variations in a)the boundary conditions, b)the fiber orientation angle θ, c)the thickness ratio h/a, and d)the number of layers *NL*, on the significance of shear deformation and degree of anisotropy are presented in Figs. 1-3.

The results of the numerical studies can be summarized as follows:

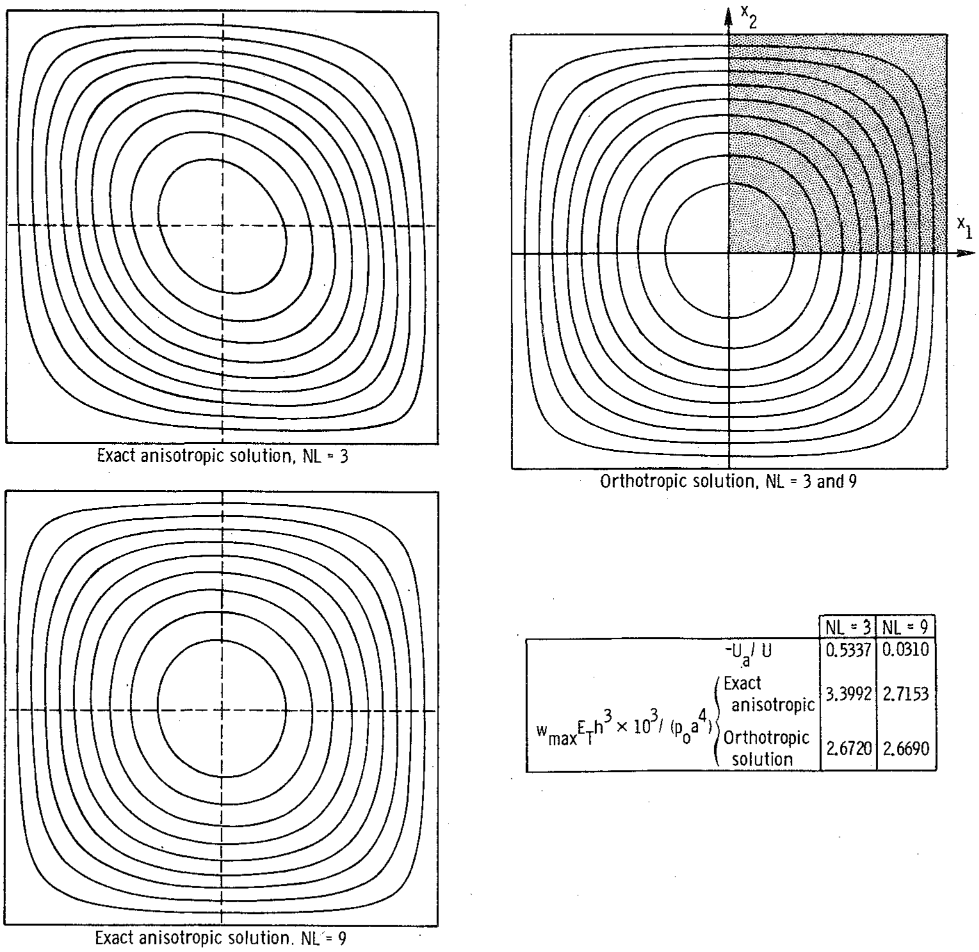
1) The shear deformation is much more pronounced in clamped plates than in simply supported plates having the same geometry and loading. The ratio U_{sh}/U for clamped plates with $h/a = 0.1$ can be as high as 0.75. For $\theta = 45°$, U_{sh}/U for clamped plates is almost twice that for simply supported plates having the same fiber orientation (see Fig. 1).

2) In single-layer plates the shear deformation decreases with the increase in the fiber orientation angle from 0 to 45°. An exception to this is the case of thin simply supported plates ($h/a \leqslant 0.01$) where the shear deformation increases with the increase of θ.

3) As the number of layers increases from 1 to 3 (or 4) the shear deformation increases. The increase being more pronounced for skew-symmetrically laminated plates. For thick plates ($h/a \geqslant 0.1$), further increase in the number of layers does not have a significant effect on the shear deformation.

4) The degree of anisotropy of the plate increases with, a) the increase in the fiber orientation angle θ (from 0 to 45°), b) the decrease in the number of layers *NL*, and c) the increase in the thickness ratio h/a for symmetrically laminated plates). Variations in the thickness ratio have more pronounced effect for clamped than for simply supported plates. Moreover, anisotropy is much more pronounced in simply supported than in clamped plates having the same geometry and loading. Figures 1 and 2 show that an increase in shear deformation is generally associated with a decrease in the degree of anisotropy and vice versa. An exception to this is the case of increasing the thickness ratio of single layer plates which results in increasing both the shear deformation and degree of anisotropy. However, the increase in the shear deformation is much more pronounced (see Fig. 3). The coupling between anisotropy and shear deformation in plates is to be contrasted with that in circular cylindrical shells where shear deformation effects were found to be more important in anisotropic than in orthotropic shells having the same geometry.[9]

5) For skew-symmetrically laminated plates the only nonzero anisotropic terms are the bending-extensional interaction coefficients B_{16} and B_{26}. Therefore the degree of anisotropy U_a/U for these plates is the same as the degree of bending-extensional coupling.



		NL = 3	NL = 9
$-U_a/U$		0.5337	0.0310
$w_{max}E_T h^3 \times 10^3/(p_0 a^4)$	Exact anisotropic	3.3992	2.7153
	Orthotropic solution	2.6720	2.6690

Fig. 4 Contour plots for normal displacement w obtained by the anisotropic (exact) and orthotropic plate theories. Simply supported symmetrically laminated plate with $h/a = 0.01$, $\theta = 30°$ subjected to uniform pressure loading.

6) Since U_a is negative, the solutions obtained by neglecting the anisotropic terms will overestimate the stiffness of the plate. Consequently, such analyses generally overestimate the buckling loads and vibration frequencies and underestimate the maximum deflections of the plate.[1,8]

7) The quantitative measures U_{sh}/U and U_a/U can be used to establish the range of validity of the classical laminated and orthotropic plate theories. This is accomplished by examining the range of the different plate parameters for which U_{sh}/U and U_a/U are small (e.g., less than 5%).

As an example to this, Fig. 4 shows contour plots for the normal displacement w of a symmetrically laminated simply supported anisotropic plate subjected to uniform loading. If the number of layers $NL = 3$, $U_a/U = -0.5337$ and the solution obtained by the orthotropic plate theory (with the anisotropic elastic coefficients neglected) is significantly in error. On the other hand, for $NL = 9$, $U_a/U = -0.0310$ and the orthotropic plate theory provides a close approximation to the exact solution (obtained by the anisotropic plate theory).

Note that while the exact solution of the three-layered plate does not exhibit the reflection type of symmetry with respect to the x_1 and x_2 axes, the exact solution of the nine-layered plate *almost* satisfies these symmetry conditions. If these "quasi" or "approximate" symmetries are utilized in the analysis of slightly anisotropic plates (with small U_a/U) and U_a is neglected, considerable reductions can result in the cost and scope of computations, with no appreciable reduction in accuracy (See Fig. 4).

In conclusion, the results of the present study show that the transverse shear flexibility and degree of anisotropy are strongly dependent on the boundary conditions, the number and stacking of the layers in addition to the degree of orthotropy of the individual layers and the thickness ratio of the plate. Moreover, the quantitative measures for the transverse shear flexibility and the degree of anisotropy suggested in this paper provide a means for establishing the range of validity of the classical laminated and orthotropic plate theories.

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